Solution of One-Speed Homogeneous Slab Transport Problems via Orthogonal Expansion of Emission Density

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INTRODUCTION

The integro-differential nature of the neutron transport equation has limited the availability of closed-form analytical solutions. Long-standing methods, such as singular eigensolutions, exist that attack this equation in its whole form [1]. Recent developments in analytical techniques by those such as Ganapol have built upon these methods for purposes of computing analytical solutions for benchmarking numerical techniques for many different problem cases [2].

In addition to the analytical methods of references [1-2], efforts to provide analytical insight have been applied to more general cases by neglecting the scattering term of the equation and computing the uncollided flux. In fact, the multiple collision method extends this approach to determine solutions to the transport equation by computing a superposition of n^{th}-collided fluxes [3-5], where the source for the n^{th}-collided flux is the n-1^{th}-collided flux. For optically thin problems without much scattering or where a recursion relation for the repeated integrals exists, this is an effective approach. However, for problems where this is not the case, the multiple collision method provides limited analytical insight.

It is in this spirit that the work of this paper takes place. In this paper, both the scattering term and the fixed source are considered together as an emission density. This emission density is then expanded in a set of known orthogonal functions. The expansion allows the angular flux to be computed as a linear superposition of analytic expressions. Unlike the multiple collision method, all terms in this superposition are computed at once rather than iteratively, and each term in the expansion is independent of the previous term. The method is demonstrated against a rigorous analytical benchmark problem.

PROBLEM SPECIFICATION

The problem studied in this paper is to determine the one-speed angular flux distribution in a homogeneous slab of length $2L$ with a beam flux of unit value incident on the left-hand boundary, a vacuum boundary condition on the right-hand boundary, and isotropic scatter. The transport equation for this problem is

$$\begin{cases} \mu \frac{\partial \psi}{\partial z} + \sigma\, \psi(z,\mu) = \frac{\sigma_s}{2}\int_{-1}^{1} d\mu'\, \psi(z,\mu') \\ \psi(-L,\mu) = \delta(\mu-\mu_0),\ \mu > 0 \\ \psi(L,\mu) = 0,\ \mu < 0 \end{cases}, \tag{1}$$

where ψ is angular flux, z is position, μ is azimuthal angle cosine, μ_0 is the azimuthal angle cosine of the incident flux, σ is total cross section, and σ_s is scatter cross section. This problem represents an analytic benchmark posed in the references [2,6]. In this paper, we proceed as in reference [6] by reframing the problem as one with vacuum boundary conditions and an interior source by solving for the diffuse, or collided, angular flux distribution. The total angular flux distribution is then given by the sum of collided and uncollided angular flux distributions, $\psi^c(z,\mu)$ and $\psi^u(z,\mu)$:

$$\psi(z,\mu) = \psi^c(z,\mu) + \psi^u(z,\mu). \tag{2}$$

The collided angular flux distribution obeys the equation

$$\begin{cases} \mu \frac{\partial \psi^c}{\partial z} + \sigma\, \psi^c(z,\mu) = \frac{\sigma_s}{2}\int_{-1}^{1} d\mu'\, \psi^c(z,\mu') + S^{eff}(z) \\ \psi^c(-L,\mu) = 0,\ \mu > 0 \\ \psi^c(L,\mu) = 0,\ \mu < 0 \end{cases}, \tag{3}$$

where $S^{eff}(z)$ is an effective volume source given by the contribution of the uncollided distribution to the scattering term:

$$S^{eff}(z) = \frac{\sigma_s}{2}\int_{-1}^{1} d\mu'\, \psi^u(z,\mu'). \tag{4}$$

For this problem, the uncollided angular flux distribution is given by

$$\psi^u(z,\mu) = \delta(\mu-\mu_0) e^{-\frac{\sigma}{\mu_0}(z+L)}, \tag{5}$$

and $S^{eff}(z)$ is given by

$$S^{eff}(z) = \frac{\sigma_s}{2} e^{-\frac{\sigma}{\mu_0}(z+L)}. \tag{6}$$

For brevity, the collided flux distribution ψ^c will be referred to as ψ for the remainder of this paper.

As mentioned previously, a common way to proceed in solving for $\psi(z,\mu)$ is to define the emission density $q(z)$ as

$$q(z) = \frac{\sigma_s}{2}\int_{-1}^{1} d\mu'\, \psi(z,\mu') + S^{eff}(z), \tag{7}$$

so that the transport problem (3) can be rewritten as (where vacuum boundary conditions have been assumed):

$$\mu \frac{\partial \psi}{\partial z} + \sigma\, \psi(z,\mu) = q(z). \tag{8}$$

Equation (8) can be solved explicitly in terms of $q(z)$. One might immediately recognize, however, that in the presence of scattering, the analytical insight provided by this equation is limited, as the emission density itself depends on the angular flux, which is unknown.

The multiple collision method is a way to remedy this. Because known quantities comprise the right-hand sides, each n^{th}-collided flux can be computed explicitly. This method has been applied with analytical techniques to generate solutions to the transport problems [4], but in the absence of a recursion relationship for the many integrations required in high scattering problems, solutions with this method are often generated numerically. As such, the method is ill-suited for solving the analytical benchmark presented.

ORTHOGONAL EXPANSION OF EMISSION DENSITY

An alternative solution method to the above is introduced here that enables a solution of the benchmark problem. At the outset we assume vacuum boundary conditions, the presence of a spatially dependent volume source $S(z)$, as in equation (3), and a definition of emission density as given by equation (7).

Suppose that the emission density is expanded in a series of complete orthogonal functions

$$q(z) = \sum_{n=0}^{\infty} a_n f_n(z). \quad (9)$$

In equation (9), $\{f_n(z)\}$ are known orthogonal functions and $\{a_n\}$ are expansion coefficients. Substituting equation (9) into equation (8), the transport equation in terms of emission density becomes

$$\mu \frac{\partial}{\partial z}\psi(z,\mu) + \sigma\, \psi(z,\mu) = \sum_{n=0}^{\infty} a_n f_n(z). \quad (10)$$

If one considers a flux mode ψ_n as the solution to

$$\mu \frac{\partial \psi_n}{\partial z} + \sigma\, \psi_n(z,\mu) = f_n(z), \quad (11)$$

then the angular flux can be computed as a superposition of these flux modes by linearity:

$$\psi(z,\mu) = \sum_{n=0}^{\infty} a_n \psi_n(z,\mu). \quad (12)$$

Since the functions f_n are known, each ψ_n is readily calculable. To determine the angular flux in equation (12), all that is left is to determine the expansion coefficients $\{a_n\}$.

To compute the expansion coefficients, the assumed form of emission density (9) is combined with its definition (7):

$$\sum_{n=0}^{\infty} a_n f_n(z) = \frac{\sigma_s}{2}\int_{-1}^{1} d\mu'\, \psi(z,\mu') + S(z). \quad (13)$$

The orthogonality of the functions $\{f_n(z)\}$ is now invoked. These functions are assumed to be orthogonal over the slab length:

$$\int_{-L}^{L} dz f_n(z) f_m(z) = w_n \delta_{nm}, \quad (14)$$

where w_n is a constant and δ_{nm} is the Kronecker delta. Both sides of equation (13) are multiplied by the m^{th}order function $f_m(z)$ and integrated over the range of orthogonality. All terms on the left-hand side of equation (13) vanish except the m^{th} term, so an expression for a_m is obtained:

$$a_m = \frac{1}{w_m}\int_{-L}^{L} dz\; f_m(z)\left\{\frac{\sigma_s}{2}\int_{-1}^{1} d\mu'\, \psi(z,\mu') + S(z)\right\}. \quad (15)$$

If one substitutes the assumed form of the angular flux given by equation (12), then equation (15) becomes, where the functional dependence of the terms has been suppressed for brevity,

$$a_m = \frac{1}{w_m}\sum_{n=0}^{\infty} a_n \int_{-L}^{L} dz \int_{-1}^{1} d\mu'\, f_m \left\{\frac{\sigma_s}{2}\psi_n + S\right\}. \quad (16)$$

The quantities Φ_{nm} and $\mathcal{S}_m$ are then defined as

$$\Phi_{nm} = \frac{\sigma_s}{2w_m}\int_{-L}^{L} dz \int_{-1}^{1} d\mu'\, f_m \psi_n, \quad (17)$$

$$\mathcal{S}_m = \frac{1}{w_m}\int_{-L}^{L} dz f_m S. \quad (18)$$

With these definitions, equation (16) can be written

$$a_m = \sum_{n=0}^{\infty} a_n \Phi_{nm} + \mathcal{S}_m. \quad (19)$$

Φ_{nm} and $\mathcal{S}_m$ are directly calculable. ψ_n is obtained from equation (11) and Φ_{nm} is computed through its definition (17) since f_m is known. Similarly, $\mathcal{S}_m$ is directly obtained through its definition (18). Therefore, equation (19) defines a system of an infinite number of linear equations for the expansion coefficients $\{a_n\}$.

To obtain a soluble form of (19), an approximation is made in the form of a truncated expansion of the emission density. If the expansion is truncated after N terms, then the system of equations (19) becomes, neglecting the error introduced by the truncation,

$$a_m = \sum_{n=0}^{N} a_n \Phi_{nm} + \mathcal{S}_m. \quad (20)$$

The system (20) can be cast in matrix form

$$(\mathbf{I} - \mathbf{\Phi})\boldsymbol{a} = \boldsymbol{\mathcal{S}}, \quad (21)$$

with $(\mathbf{I} - \mathbf{\Phi})$ as an $(N \text{ x } N)$ matrix, $\boldsymbol{a}$ and $\boldsymbol{\mathcal{S}}$ as $(N \text{ x } 1)$ vectors, and

$$\begin{cases} [\mathbf{I} - \mathbf{\Phi}]_{nm} = \delta_{nm} - \Phi_{nm} \\ [\boldsymbol{a}]_n = a_n \\ [\boldsymbol{\mathcal{S}}]_n = \mathcal{S}_n \end{cases}. \quad (22)$$

The transport problem given by equation (3) is solved by determining the expansion coefficients from equation (21) and computing the angular flux through equation (12).

APPLICATION OF THE METHOD WITH FOURIER SERIES

Thus far, the only stipulations on the functions $\{f_n(z)\}$ are that their range of orthogonality is the size of the slab and that they are complete. While there are many choices for orthogonal functions, Fourier series are chosen for this paper:

$$q(z) = \sum_{n=0}^{\infty}\left\{a_n \cos\left(\frac{n\pi z}{L}\right) + b_n \sin\left(\frac{n\pi z}{L}\right)\right\}. \quad (23)$$

Fourier series are chosen because of their simplicity in implementation. Only two analytical and semi-analytical expressions are needed to determine all flux modes and matrix elements, respectively, computed in seeking solutions. This choice requires a slight modification of the method discussed previously. With Fourier series, the angular flux distribution is constructed as

$$\psi(z,\mu) = \sum_{n=0}^{\infty}\{a_n \psi_n^C(z,\mu) + b_n \psi_n^S(z,\mu)\}, \quad (24)$$

where all ψ_n^C and all ψ_n^S are be determined from the solution to equations (25) and (26), respectively:

$$\mu \frac{\partial \psi_n^C}{\partial z} + \sigma \psi_n^C(z,\mu) = \cos(\frac{n\pi z}{L}), \quad (25)$$

$$\mu \frac{\partial \psi_n^S}{\partial z} + \sigma \psi_n^S(z,\mu) = \sin(\frac{n\pi z}{L}). \quad (26)$$

These equations were solved analytically. ψ_n^C and ψ_n^S are given by equations (27) and (28), respectively:

$$\psi_n^C = \begin{cases} \frac{\sigma L^2 \cos\frac{n\pi z}{L} + \pi\mu n L \sin\frac{n\pi z}{L} - \sigma L^2 (-1)^n e^{\frac{-\sigma(L+z)}{\mu}}}{[(\sigma L)^2 + (n\pi\mu)^2]} \mu > 0 \\ \frac{\sigma L^2 \cos\frac{n\pi z}{L} + \pi\mu n L \sin\frac{n\pi z}{L} - \sigma L^2 (-1)^n e^{\frac{\sigma(L-z)}{\mu}}}{[(\sigma L)^2 + (n\pi\mu)^2]} \mu < 0 \end{cases}, \quad (27)$$

$$\psi_n^S = \begin{cases} \frac{\sigma L^2 \sin\frac{n\pi z}{L} - \pi\mu n L \cos\frac{n\pi z}{L} + \pi\mu n L (-1)^n e^{\frac{-\sigma(L+z)}{\mu}}}{(\sigma L)^2 + (n\pi\mu)^2} \mu > 0 \\ \frac{\sigma L^2 \sin\frac{n\pi z}{L} - \pi\mu n L \cos\frac{n\pi z}{L} + \pi\mu n L (-1)^n e^{\frac{\sigma(L-z)}{\mu}}}{(\sigma L)^2 + (n\pi\mu)^2} \mu < 0 \end{cases}. \quad (28)$$

Since two different expressions give ψ_n^C and ψ_n^S, two expressions are needed for the matrix elements Φ_{nm}. Equation (29) gives the semi-analytic matrix elements Φ_{nm}^C corresponding to ψ_n^C, and equation (30) gives the semi-analytic matrix elements Φ_{nm}^S corresponding to ψ_n^S:

$$\Phi_{nm}^C = \frac{\sigma L^2}{w_n} \int_0^1 d\mu \frac{w_n \delta_{nm} - \frac{(-1)^{n+m} \mu \sigma L^2 (1 - e^{\frac{-2\sigma L}{\mu}})}{\sigma^2 L^2 + \pi^2 \mu^2 m^2}}{\sigma^2 L^2 + \pi^2 \mu^2 n^2}, \quad (29)$$

$$\Phi_{nm}^S = \frac{1}{w_n} \int_0^1 d\mu \frac{\sigma L^2 w_n \delta_{nm} - \frac{(-1)^{n+m} \mu n n' (\pi\mu L)^2 (1 - e^{\frac{-2\sigma L}{\mu}})}{\sigma^2 L^2 + \pi^2 \mu^2 m^2}}{\sigma^2 L^2 + \pi^2 \mu^2 n^2}. \quad (30)$$

Similarly, two expressions are needed for the values $\{\mathcal{S}_n\}$. Equation (31) gives $\mathcal{S}_n^C$, corresponding to ψ_n^C. Equation (32) gives $\mathcal{S}_n^S$, corresponding to ψ_n^S. The source used is the effective source for the benchmark given in equation (6).

$$\mathcal{S}_n^C = \frac{\sigma_s}{2w_n} \frac{(-1)^n \mu_0 \sigma L^2 (1 - e^{\frac{-2\sigma L}{\mu_0}})}{\sigma^2 L^2 + \pi^2 \mu_0^2 n^2}, \quad (31)$$

$$\mathcal{S}_n^S = \frac{\sigma_s}{2w_n} \frac{(-1)^n \pi \mu_0^2 n L (1 - e^{\frac{-2\sigma L}{\mu_0}})}{\sigma^2 L^2 + \pi^2 \mu_0^2 n^2}. \quad (32)$$

The constants $\{w_n\}$ are given by

$$w_n = \begin{cases} 2L, n = 0 \\ L, n > 0 \end{cases}. \quad (33)$$

Equations (29), (31), and (33) are used to construct a system of equations of the form of equation (21) for the coefficients $\{a_n\}$. Equations (30), (32), and (33) are used to construct a system of equations of the form of equation (21) for the coefficients $\{b_n\}$. Once these coefficients are computed, the angular flux distribution can be determined.

NUMERICAL IMPLEMENTION AND RESULTS

The expansion method was implemented as described above with analytic flux modes to determine the collided flux distribution. In accordance with published benchmark results, $\sigma = 1$, $\sigma_s = 0.9$, left-hand incident flux was assumed to be normal ($\mu_0 = 1$), and the total thickness of the slab was assumed to be one mean free path ($2L = 1$). With these parameters, the angular flux distribution was evaluated on the left-hand side of the slab, called the reflected flux, and the right-hand side of the slab, called the transmitted flux. In evaluating the semi-analytic matrix elements, the angular integrals were evaluated with composite Simpson's rule quadrature with 200 subintervals.

Included in the numerical results were multiple evaluations of the angular flux at $\mu = 0$ that required special consideration. Following the approach of reference [7], the angular flux distribution for this case was determined from the transport equation evaluated at $\mu = 0$. Equation (34) gives the expression for the angular flux for the $\mu = 0$ case with the appropriate effective source and flux mode substitutions being made, where again functional dependencies are suppressed for brevity:

$$\psi = \frac{\sigma_s}{2\sigma} \left\{ \int_{-1}^{1} d\mu' \sum_{n=0}^{N} [a_n \psi_n^C + b_n \psi_n^S] + S^{eff} \right\}. \quad (34)$$

The flux values for equation (34) are semi-analytic, and the angular integral was evaluated with composite Simpson's rule quadrature with 10000 subintervals.

The reflected and transmitted flux were evaluated with increasing values of expansion order N. The results for the reflected flux are given in Table 1. The results for the transmitted flux are given in Table 2.

Table 1. The reflected angular flux for increasing expansion order.

μ	N = 200	N = 2000	N = 20000
-1.0	4.19946E-01	4.19996E-01	4.20001E-01
-0.9	4.47704E-01	4.47763E-01	4.47769E-01
-0.8	4.78865E-01	4.78936E-01	4.78943E-01
-0.7	5.13851E-01	5.13937E-01	5.13946E-01
-0.6	5.52970E-01	5.53078E-01	5.53088E-01
-0.5	5.96150E-01	5.96287E-01	5.96301E-01
-0.4	6.42331E-01	6.42513E-01	6.42531E-01
-0.3	6.88198E-01	6.88452E-01	6.88478E-01
-0.2	7.26175E-01	7.26568E-01	7.26608E-01
-0.1	7.44457E-01	7.45250E-01	7.45329E-01
0.0	7.18429E-01	7.18702E-01	7.18736E-01

Table 2. The transmitted angular flux for increasing expansion order.

μ	N = 200	N = 2000	N = 20000
1.0	3.80585E-01	3.80535E-01	3.80530E-01
0.9	4.01474E-01	4.01415E-01	4.01409E-01
0.8	4.23845E-01	4.23774E-01	4.23767E-01
0.7	4.47363E-01	4.47277E-01	4.47268E-01
0.6	4.71192E-01	4.71084E-01	4.71074E-01
0.5	4.93511E-01	4.93373E-01	4.93360E-01
0.4	5.10607E-01	5.10425E-01	5.10407E-01
0.3	5.15447E-01	5.15192E-01	5.15167E-01
0.2	4.97093E-01	4.96700E-01	4.96660E-01
0.1	4.48947E-01	4.48155E-01	4.48076E-01
0.0	3.72589E-01	3.72316E-01	3.72282E-01

The angular flux was also computed within the slab for a variety of values of μ. Forward-directed ($\mu \geq 0$) angular flux distributions are given in Figure 1. Backward-directed ($\mu \leq 0$) angular flux distributions are given in Figure 2.

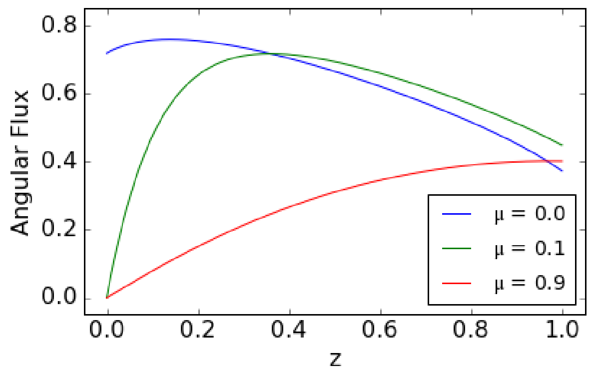

Figure 1. Forward-directed angular flux distributions for a variety of values of μ.

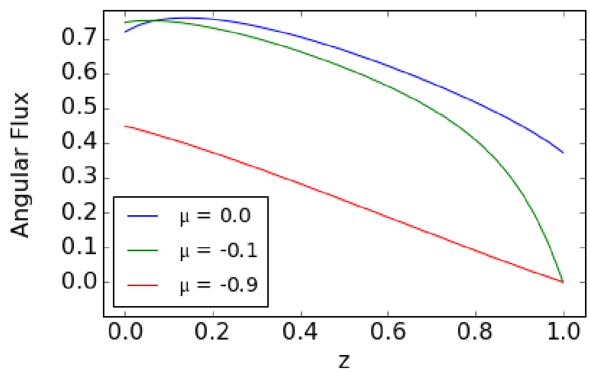

Figure 2. Backward-directed angular flux distributions for a variety of values of μ.

For the highest order cases, the computed angular flux values have a relative difference of 10^{-5} or less from the benchmark values of reference [2] for all cases except μ = -0.1 for the reflected angular flux and μ = 0.0 and μ = 0.1 for the transmitted case. In all cases, the angular flux values for N = 20000 have an absolute difference of less than 10^{-5} from the benchmark values of reference [2]. Generally, the expansion order needed to increase by an order of magnitude to reduce error (relative or absolute) by an order of magnitude. From this, it can be concluded that the apparent error of the method is of order $O\left(\frac{1}{N}\right)$. This numerical demonstration shows that the method introduced in this paper can be used to solve analytic benchmarks existing in literature where a similar method, the multiple collision method, would be inappropriate.

CONCLUSION AND FUTURE WORK

The method introduced in this paper allows for angular flux distributions to be represented as a superposition of analytical expressions. Unlike the multiple collision method, the expansion method of this paper features flux modes that are computed independently of one another so that there is no reliance upon a recurrence relationship between collided flux generations or upon evaluating many integrals to obtain an analytic result. The method was applied to evaluate reflected and transmitted fluxes for an established analytical benchmark.

Future work will involve extending the expansion method's applicability. This will involve analyzing and improving the convergence behavior of the method as well as generalizing the method to other physical situations. In these more general cases, emission density is possibly angularly dependent, which will require a more sophisticated expansion. Further, problems with anisotropic scatter will be considered, which will require generalizing the matrix elements Φ_{nm} to include anisotropic scattering cross sections.

It would also be of interest to compare this method to other expansion methods that solve the transport equation via direct expansion of the flux rather than the emission density, such as the work of reference [8].

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