A New Stable Finite Element Method for Solving the First-Order Neutron Transport Equation

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INTRODUCTION

Finite element method (FEM) has become a popular choice to solve the neutron transport equation (NTE) due to its good geometric adaptability. The standard Continuous-Galerkin (CG) method is mainly used in second order equation. The even-parity (EP) equation and the self-adjoint angular flux (SAAF) equation have received much attention in the past few years. But neither of them can accurately describe void region because the total cross section appears in the denominator.

Applying Galerkin method to the first-order neutron transport equation tends to produce spurious oscillations. In order to obtain robust solution, some stabilization technique have been developed [1], such as Streamline-Upwinding Petrov-Galerkin (SUPG) method, Galerkin Least-Square (GLS) method, etc. These methods require a stabilization term added to its test functions which will incur more computational cost. Moreover, the selection of free parameter for SUPG method is indeterminacy and the boundary condition for GLS method is difficult to deal with. Another finite element method that has been widely used is Discontinuous-Galerkin (DG) method, its main disadvantage is the number of unknowns is much larger than CG method. For tetrahedron element, the number of unknowns is about 22 times larger.

This summary presents a new stable finite element method for solving the first-order neutron transport equation based on sub-grid scale (SGS) discretization. SGS method was first applied to neutron transport equation in 2010 by A. G. Buchan[2], and an efficient Sub-grid scale (ESGS) method [3] was proposed in 2016 which decouples the discontinuous components of angular moments partly by the angular moments grouping. This summary presents a method which can decouple all the angular moments of the discontinuous components, and what's more, the starting point of this method is different from the original SGS method, it provides a new idea to construct the stable finite element method framework for the first order equation.

THEORY

In this summary, We solve the monoenergetic, steady-state transport equation:

$$\boldsymbol{\Omega}\cdot\nabla\phi(\boldsymbol{r},\boldsymbol{\Omega})+\sigma_t(\boldsymbol{r})\phi(\boldsymbol{r},\boldsymbol{\Omega}) = \int_{\boldsymbol{\Omega}'}\sigma_s(\boldsymbol{r},\boldsymbol{\Omega}'\cdot\boldsymbol{\Omega})\phi(\boldsymbol{r},\boldsymbol{\Omega}')d\boldsymbol{\Omega}'+s(\boldsymbol{r},\boldsymbol{\Omega}), \quad (1a)$$

inside $\boldsymbol{r}\in V$, $\boldsymbol{\Omega}\in 4\pi$, with an incident boundary condition

$$\phi(\boldsymbol{r},\boldsymbol{\Omega})=g(\boldsymbol{r},\boldsymbol{\Omega}), \quad \boldsymbol{r}\in\partial V,\ \boldsymbol{\Omega}\cdot\boldsymbol{n}<0. \quad (1b)$$

For simplicity, we move the scattering source to the left, and rewrite Eq. (1) as an operator form:

$$\boldsymbol{\Omega}\cdot\nabla\phi+\sigma\phi=s. \quad (2)$$

where the first term in the left side represents leakage rate or convection, the second term in the left side represents reaction rate, the right hand side represents source.

Define the inner product form as follows:

$$(f,g)=\int_V\int_{\boldsymbol{\Omega}} fgd\boldsymbol{\Omega}d\boldsymbol{r}, \quad (3a)$$

$$\langle f,g\rangle=\int_{\partial V}\int_{\boldsymbol{\Omega}}\boldsymbol{\Omega}\cdot\boldsymbol{n}fgd\boldsymbol{\Omega}d\boldsymbol{r}, \quad (3b)$$

$$\langle f,g\rangle^+=\int_{\partial V}\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}>0}\boldsymbol{\Omega}\cdot\boldsymbol{n}fgd\boldsymbol{\Omega}d\boldsymbol{r}, \quad (3c)$$

$$\langle f,g\rangle^-=\int_{\partial V}\int_{\boldsymbol{\Omega}\cdot\boldsymbol{n}<0}\boldsymbol{\Omega}\cdot\boldsymbol{n}fgd\boldsymbol{\Omega}d\boldsymbol{r}. \quad (3d)$$

Applying Galerkin method to Eq. (2), that multiplies a test function φ in both sides

$$(\varphi,\boldsymbol{\Omega}\cdot\nabla\phi+\sigma\phi)=(\varphi,s). \quad (4)$$

Integrate it by parts to convert the volumetric integral to surface integral,

$$\langle\varphi,\phi\rangle-(\boldsymbol{\Omega}\cdot\nabla\varphi,\phi)+(\varphi,\sigma\phi)=(\varphi,s). \quad (5)$$

The above is Continuous-Galerkin (CG) method. The mainly difference between SGS method with CG method is that the SGS method divides the unknow into two parts, one represents the continuous part and the other represents the residual in each element.

$$\phi=\phi_c+\phi_r. \quad (6)$$

Inserting Eq. (6) to Eq. (4), and Eq. (5), and the Eq. (4) integrate in an element scale, Eq. (5) integrate in whole scale,

$$(\varphi,\boldsymbol{\Omega}\cdot\nabla\phi_c)_e+(\varphi,\boldsymbol{\Omega}\cdot\nabla\phi_r)_e+(\varphi,\sigma\phi_c)_e+(\varphi,\sigma\phi_r)_e=(\varphi,s)_e, \quad (7a)$$

$$\langle\varphi,\phi\rangle-(\boldsymbol{\Omega}\cdot\nabla\varphi,\phi_c)-(\boldsymbol{\Omega}\cdot\nabla\varphi,\phi_r)+(\varphi,\sigma\phi_c)+(\varphi,\sigma\phi_r)=(\varphi,s). \quad (7b)$$

In each element, we assume $\boldsymbol{\Omega}\cdot\nabla\phi_d=c\phi_d$, the coefficient c is indeterminacy, currently, we let it equal to 1.0, but some other options can also get the right results. The boundary condition was applied to Eq. (7b),

$$(\varphi,\boldsymbol{\Omega}\cdot\nabla\phi_c)_e+(\varphi,\sigma\phi_c)_e+(\varphi,(\sigma+c)\phi_r)_e=(\varphi,s)_e, \quad (8a)$$

$$\langle\varphi,\phi\rangle^+-(\boldsymbol{\Omega}\cdot\nabla\varphi,\phi_c)-(\boldsymbol{\Omega}\cdot\nabla\varphi,\phi_r)+(\varphi,\sigma\phi_c)+(\varphi,\sigma\phi_r)=(\varphi,s)-\langle\varphi,g\rangle^-. \quad (8b)$$

The Eq. (8) is the final variational form of the new stable finite element method, which we can solve the ϕ_r from Eq. (8a) and then insert it to Eq. (8b) to solve the ϕ_c.

For Galerkin method, the unknows are expended by basis functions, there are two sets of basis functions, spatial and angular basis functions. Choosing the linear piecewise polynomials as the spatial basis functions which are denoted by $\boldsymbol{N}(\boldsymbol{r})$, and the spherical harmonics as the angular basis functions which are denoted by $\boldsymbol{Y}(\boldsymbol{\Omega})$, the whole basis functions can be expressed as the tensor product of spatial and angular basis functions. The test functions are basis functions, so

$$\boldsymbol{\varphi}(\boldsymbol{r},\boldsymbol{\Omega}) = \boldsymbol{N}(\boldsymbol{r}) \otimes \boldsymbol{Y}(\boldsymbol{\Omega}) . \tag{9}$$

The unknowns can be expanded by basis functions

$$\phi_c(\boldsymbol{r},\boldsymbol{\Omega}) = \boldsymbol{\varphi}(\boldsymbol{r},\boldsymbol{\Omega})^T \boldsymbol{\phi}_c , \tag{10a}$$

$$\phi_r(\boldsymbol{r},\boldsymbol{\Omega}) = \boldsymbol{\varphi}(\boldsymbol{r},\boldsymbol{\Omega})^T \boldsymbol{\phi}_r . \tag{10b}$$

Inserting Eq. (10) to Eq. (8), and define the following vector and matrix

$$\boldsymbol{s}_c = (\boldsymbol{\varphi}, s) - \langle \boldsymbol{\varphi}, g \rangle^- , \tag{11a}$$

$$\boldsymbol{s}_r = (\boldsymbol{\varphi}, s)_e , \tag{11b}$$

$$\mathbf{A} = \left\langle \boldsymbol{\varphi}, \boldsymbol{\varphi}^T \right\rangle^+ - \left(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\varphi}^T \right) + \left(\boldsymbol{\varphi}, \sigma \boldsymbol{\varphi}^T \right) , \tag{11c}$$

$$\mathbf{B} = -\left(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\varphi}^T \right) + \left(\boldsymbol{\varphi}, \sigma \boldsymbol{\varphi}^T \right) , \tag{11d}$$

$$\mathbf{C}_e = \left(\boldsymbol{\varphi}, \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}^T \right) + \left(\boldsymbol{\varphi}, \sigma \boldsymbol{\varphi}^T \right) , \tag{11e}$$

$$\mathbf{D}_e = \left(\boldsymbol{\varphi}, (\sigma + c) \boldsymbol{\varphi}^T \right)_e . \tag{11f}$$

From the above, the residual $\boldsymbol{\phi}_r$ can be solved from Eq. (8a),

$$\boldsymbol{\phi}_{re} = \mathbf{D}_e^{-1} (\boldsymbol{s}_r - \mathbf{C}_e \boldsymbol{\phi}_c) . \tag{12}$$

In every element, inserting Eq. (12) to Eq. (8b) yields,

$$(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})\boldsymbol{\phi}_c = \boldsymbol{s}_c - \mathbf{B}\mathbf{D}^{-1}\boldsymbol{s}_d . \tag{13}$$

Using lumping technique to matrix $\mathbf{D}_e$ that adds every element in the row to the diagonal and then set the rest to zero, which means the element i of the lumping matrix $\mathbf{D}_l$ is

$$[\mathbf{D}_l]_i = \sum_{j=1}^{J} [\mathbf{D}_e]_{ij} = [(\boldsymbol{\varphi}, \sigma + c)_e]_i . \tag{14}$$

Inserting Eq. (14) to Eq. (13) and then solving it obtained the final results $\boldsymbol{\phi}_c$.

RESULTS

The stability and void applicability of this new finite element framework was tested by an analytical problem of pure absorption materials, and the Reed Cell benchmark, and the Maynard-Watanabe problem.

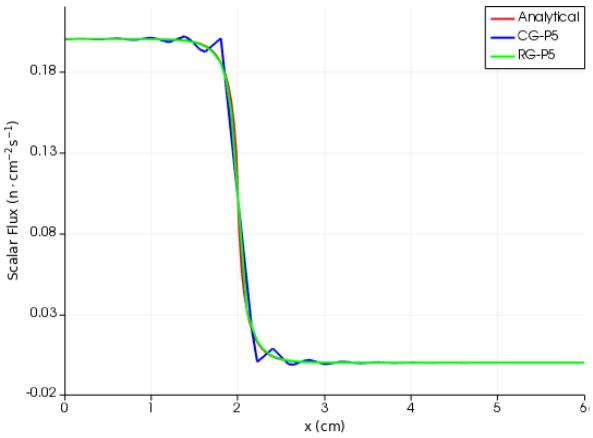

Fig. 1: The results of analytical problem.

Analytical problem

The analytical problem is a one-dimensional problem, the total length is 6 cm, the source region is located at [0,2], and its intensity equal to $1\text{n} \cdot \text{cm}^{-2}\text{s}^{-1}$, the absorption cross section $\sigma_a = 5.0$. The results are as follows

The Fig. 1 contains the analytical result and the Continuous-Galerkin result (CG-P5), and the new method (I name this Residual-Galerkin method) result (RG-P5). The oscillation phenomenon of CG result can be found obviously, and the solution contains negative scaler flux, while the RG result is in good agreement with analytical solution.

Reed cell benchmark

The second problem is Reed cell benchmark. It is a one dimensional problem that contains 5 regions with a total length of 8 cm, the length of each region are 2 cm, 1 cm, 2 cm, 1 cm, 2 cm. and the material properties are given in Table. I.

TABLE I: Cross Section of Reed Cell Problem

Region	σ_t/cm^{-1}	σ_s/cm^{-1}	$s/\text{cm}^{-2} \cdot \text{s}^{-1}$
1	50.0	0.0	50.0
2	5.0	0.0	0.0
3	0.0	0.0	0.0
4	1.0	0.9	1.0
5	1.0	0.0	0.0

The results are shown in Fig. 2. The CG result also exhibits the unfavorable oscillation phenomenon, especially in the pure absorption and void region. The RG results are stable, and with the increase of P_N order, the reference solution is approximated gradually.

Maynard Watanabe benchmark

The third problem is a Maynard-Watanabe benchmark. It is a two-dimensional problem which contains inner vacuum. The geometry is shown in Fig. 3, and the material properties are given in Table. II.

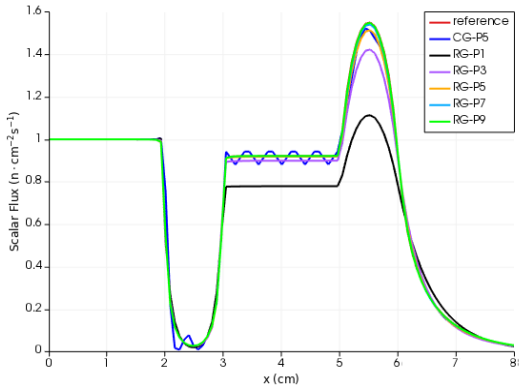

Fig. 2: Reed cell problem.

TABLE II: Cross Section of Maynard Watanabe Problem

Region	σ_t/cm^{-1}	σ_s/cm^{-1}	$s/\text{cm}^{-2}\cdot\text{s}^{-1}$
1	0.2	0.19	5.4
2	0.0	0.0	0.0
3	0.2	0.19	0.0

Figures 4 and 5 show the scaler flux distribution on the whole spatial domain and along the dash line of Fig. 3, respectively.

CONCLUSIONS

In this summary, a new stable finite element method framwork for solving the first-order neutron transport equation has been established. The new method has many advantages. Firstly, it belongs to continuous galerkin method which the number of unknows are much less than discontinuous galerkin method; Secondly, The computation cost is much less than traditional SGS method and Petrov-Galerkin method; Thirdly, in addition to Petrov-Galerkin method, this summary developed a totaly new idea to construct the stable finite element method for the first-order hyperbolic equation. Further more, there is a lot of potential of this method to be explored.

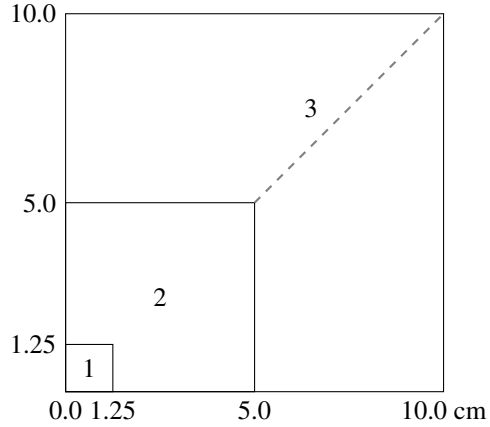

Fig. 3: Geometry of Maynard Watanabe problem

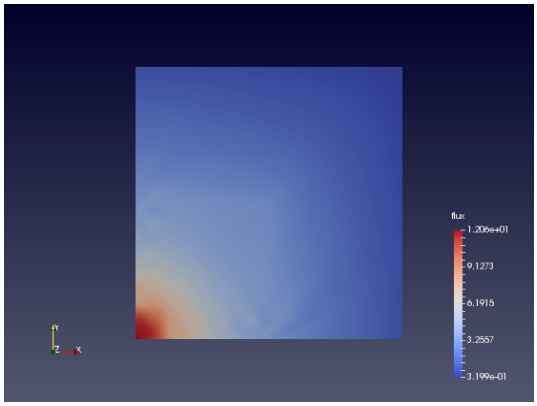

Fig. 4: Distribution of scaler flux.

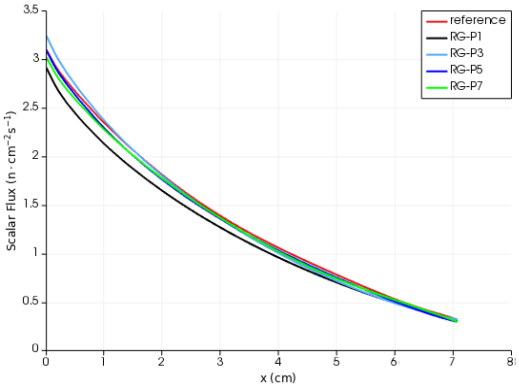

Fig. 5: The scalar flux along the dash line.

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